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Abstract. This work provides a mathematical analysis of a thermo-diffusive combustion model for lean spray flames. We prove that the large activation energy limit of the model enjoys explicit expressions of two types correponding to two distinct combustion regimes, in contrast with similar gaseous flames as studied in [4]. The transition parameter is proved to be the ratio between the speed of vaporisation and the speed of combustion, and we are able to prove and estimate the thickening of the combustion zone in the case of slow vaporisation, which is again a novelty compared to purely gaseous combustion. The model at task is therefore on the one hand simple enough to allow for explicit asymptotic limits, and on the other hand rich enough to capture some particular aspects of spray combustion. Finally, we briefly consider situations where the vaporisation is infinitely fast, where the spray is polydisperse, or where the geometry of the problem is different from that of the onedimensional travelling wave.

1. Introduction

This paper provides a rigourous mathematical analysis of some aspects of spray combustion, including the analysis of the so-called large activation energy limit for a spray flame model. This notion of large activation energy limit was first introduced in the pioneering work of Zeldovich and Frank-Kamenetskii [27], and refers to the limit where the combustion rate is much faster than any other physical phenomenon, in particular diffusion. Gas-vapor-droplets systems have many applications in industry or everyday's life, such as diesel or propulsion engines. When trying to understand some specific features, it appears that the structure of these two-phase flames as well as their speed or stability are greatly affected by the presence of vaporising liquid droplets possibly interacting with the combustion zone.

The behaviour of spray flames has been investigated a lot in the physics literature and a wide variety of regimes were considered. Dating from the 70s and early 80s, we can quote the works of Polymeropoulos et al. [21][22], Mizutani et al. [19][20], Hayashi et al. [15][16]. Ballal et al. [2]. They present studies of the propagation of liquid fuel sprays, study the influence of the size of the droplets, the type and geometry of the spray flame as well as its structure. The investigation of those vapor–drop–air systems was continued in the 90s. To quote but a few works, we refer to the work of Aggarwal and Sirignano [1] as well as the papers of Greenberg, Tambour and Silverman [23][24], where the structure of spray flames as well as the influence of parameters such as droplet size, fuel volatility, or equivalence ratio are also investigated analytically. More elaborate situations appearing in propulsion engines are for example pulsating or acoustic instabilities. We refer to [14][13][9][10][11][7] and more recently [8][12][17][18] for studies in that direction.

Only few of these studies of spray flames (e.g. [8]) involve rigorous–in–the–mathematical–sense analysis of the model at task where existence, uniqueness, or asymptotic limits are derived. This is in contrast with purely gaseous combustion, where a lot of results exist in the mathematical literature for various regimes and asymptotics. In particular, a complete study of thermo-diffusive lean gaseous flame fronts and the large activation energy limit is in the paper of Berestycki, Nicolaenko and Scheurer [4], and in the paper of Berestycki and Larrouturou [3]. The originality of the present work is to provide a complete mathematical analysis of the counterpart of those systems, namely a lean **spray** flame model that on the hand is **simple enough** to allow for explicit asymptotic limits, and on the other hand **rich enough** to capture some particular aspects of spray combustion. We prove in particular the existence of **two distinct combustion regimes** in the large activation energy limit. We prove also that the speed of the spray flame decreases when the time for complete vaporisation of the droplets exceeds a critical value. Those results should be compared with the very interesting work of Suard, Nicoli and Haldenwang [25]. Using numerical simulations, these authors investigate the scaling laws of the spray flame velocity versus the Damkhoeler number (i.e. the ratio of the typical time for vaporisation versus the typical time for combustion). They also studied the structure of the reaction zone, of which they prove that the complexity increases in the so–called "vaporisation–controlled regime". Our mathematical analysis shows similar results, with however a sharp transition in the large activation energy limit. Also, an important result of the present work is that we can characterize completely the limiting profiles, with **explicit** expressions.

The model we consider is built as an elaboration of the purely gaseous thermo-diffusive lean flame (see [4],[3]) that originally involves only the temperature T of the mixture and the fraction Y of the gaseous fuel. We couple this thermo-diffusive model to vaporising droplets that act as a reservoir of liquid fuel. That is, the vaporisation of the liquid droplets produces (part of) the fuel vapor needed for the combustion. There is no combustion of the liquid phase. This leads to the following minimal model for a lean spray flame:

$$\begin{cases} -T''+cT' &= f(T)Y, \\ -dY''+cY' &= -f(T)Y+2\pi n\, g(T)\sqrt{R}, \\ cR' &= -g(T)H(R), \end{cases} \tag{1}$$

supplemented with conditions at infinity

$$\begin{cases} T(-\infty) = 0, & T(+\infty) = T_b, \\ Y(-\infty) = Y_u, & Y(+\infty) = 0, \\ R(-\infty) = R_u. \end{cases}$$

where $T(x,\, t)$ is the nondimensional temperature, $Y(x,\, t)$ the fraction of reactant, $R(x,\, t)$ the square radius (i.e. $4\,\pi\, R$ is the surface of the droplet), and H is the Heaviside function. Also, we have $T_b := Y_u + 4/3\,\pi\, n_u\, R_u^{3/2}$. Finally, $f(T)$ is the reaction rate and $g(T)$ is the vaporisation rate. The reaction term is of ignition type, that is f is nonnegative, piecewise Lipschitz continuous, and there exists an ignition temperature $\theta_i > 0$ such that $f(T) \equiv 0$ for $T \le \theta_i$ and $f(T) > 0$ for $T > \theta_i$. The function g satisfies exactly the same hypotheses but with a boiling temperature $\theta_v < \theta_i$, which means that the droplets start vaporising before the combustion starts. Finally, the constant d is the inverse of the Lewis number. At constrast with the classical thermo-diffusive system for gas flames, the system above in the case $d = 1$ does not reduce to a single scalar equation. Finally, notice that this model does not include the effect of the latent heat. The latent heat would appear as a loss term proportional to the vaporisation term in the equation for the temperature. Neglecting this phenomena is certainly an important simplification in the case where the latent heat is large. We therefore assume implicitely in thius study that this effect is negligible.

The plan of the paper is as follows. Section 2 is devoted to the proof of the existence of travelling waves for the system (1) above as stated in Theorem 1 for the monodisperse case, that is in the case where the droplets at a given position all have the same size. The extension to the case of polydisperse sprays is briefly discussed in section 4.2. Section 3 is devoted to the study of the high activation energy limit, where one assumes that combustion is an infinitely fast process. Suitable scalings for the reaction rate give rise to a limiting problem where combustion occurs in a possibly infinitely small region:

$$\begin{cases} -T''+c\,T' &= cT_b\delta_{x=0}, \\ -d\,Y''+c\,Y' &= -cT_b\delta_{x=0}+2\,\pi\, n_u\, g(T)\sqrt{R}, \\ c\,R' &= -g(T)\,H(R), \end{cases} \quad \begin{cases} T(-\infty)=0, & T(\infty)=T_b \\ Y(-\infty)=Y_u, & Y(\infty)=0 \\ R(-\infty)=R_u, \end{cases} \tag{2}$$

As announced before, we will prove that this system obeys two different regimes depending on the size of the droplets (i.e. depending on the Damkhoeler number). With the notations above, when the size of the droplets is below a critical size R_c, the droplets finish vaporising *before* the reaction front, and the velocity of the flame is a constant equal to the velocity of the equivalent gas flame. On the contrary, when the size of the droplets is larger than the critical value R_c, the velocity of the flame starts decreasing. This is the so-called "vaporisation-limited" regime (see [25]). Moreover, by zooming on the reaction zone in the large activation energy limit, we are able to estimate the size of the vaporisation region that intersects the combustion region. These results are stated in Theorem 11.

2. Existence of Travelling Waves

We want to prove here

Theorem 1. *There exists at least one travelling wave* (T, Y, R, c) *in* $X = \mathcal{C}^1(\mathbb{R}) \times \mathcal{C}^1(\mathbb{R}) \times \mathcal{C}(\mathbb{R}) \times \mathbb{R}$ *solution to Problem (1) above .*

The study of the existence of travelling waves for system (1) is done following the strategy of H. Berestycki, B. Nicolaenko and B. Scheurer [4]. Namely : in **Step 1**, we state the problem in a bounded domain. In **Step 2** we rewrite the problem as a fixed point problem involving an homotopy parameter. In **Step 3**, we prove *a priori* estimates *independent* of the size of the domain. In **Step 4**, we prove the existence of a travelling wave in a bounded domain. Finally, in **Step 5**, we let the size of the domain go to infinity. The major step is **Step 3** whose proof is postponed to the next section. The proof of **Step 5** being very similar to the classical gaseous flame case, we will not detail it here (see [4]).

Steps 1 & 2: Problem in a bounded domain & homotopy technique Let a be a fixed positive number. We consider the following boundary value problem on $[-a, a]$:

Problem 2. *Find* (T, Y, R, c) *satisfying in* $]-a, a[$*:*

$$\left\{\begin{aligned} -T'' + c\,T' &= \tau\,[f(T)\,Y], \\ -d\,Y'' + c\,Y' &= \tau\left[-f(T)\,Y + 2\,\pi\,n\,g(T)\sqrt{R}\right], \\ c\,R' &= \tau\,[-g(T)\,H(R)], \\ c &= T(0) - \theta_i + c \end{aligned}\right. \tag{3}$$
$$\left\{\begin{aligned} &-T'(-a) + c\,T(-a) = 0, && T(a) = T_b, \\ &-d\,Y'(-a) + c\,T(-a) = Y_u, && Y(a) = 0, \\ &R(-a) = R_u, \end{aligned}\right.$$

We will note $X_a = \mathcal{C}^1([-a, a]) \times \mathcal{C}^1([-a, a]) \times \mathcal{C}([-a, a]) \times \mathbb{R}$. Now, injecting functions (T, Y, R, c) in the Right-Hand-Side of the system, integrating and solving for (T, Y, R, c) in the Left-Hand-Side defines a fixed point problem that we denote by $F_\tau \equiv I - K_\tau$, so that proving the existence of a solution (T, Y, R, c) to system (3) is equivalent to finding (T, Y, R, c) such that $F_\tau(T, Y, R, c)$ cancels. Thanks to the boundedness of the domain and the regularizing properties of the elliptic operator, one verifies easily that we have:

Lemma 3. *Assume that the functions f and g are continous and Lipshitz. The mapping F: $X_a \times [0, 1] \longrightarrow X_a$ defined by $(T, Y, R, c) \times \tau \longmapsto (I - K_\tau)(T, Y, R, c)$ is well-defined, compact and uniformly continous with respect to τ.* □

Step 3: Uniform *a priori* estimates The following result is proved in Section 2.1 below:

Lemma 4. *Let F_τ be defined as above. There exists finite positive constants M, $\underline{c}$, $\bar{c}$ independent of a, such that, for Ω defined as:*

$$\Omega := \{(T, Y, R, c) \in X \quad ; \quad \|T\|_{\mathcal{C}^1(I_a)} \leqslant M,\ \|Y\|_{\mathcal{C}^1(I_a)} \leqslant M,\ \|R\|_{\mathcal{C}(I_a)} \leqslant M,\ \underline{c} < c < \bar{c}\}$$

Step 4: Application of the Leray-Schauder Topologocal Degree. Assuming the uniform *a priori* estimates above, we can find positive constants $M, \underline{c}, \bar{c}$ such that the solution of the fixed point cannot be found on the frontier $\partial\Omega$. Also, using Lemma 3, we have:

$$\begin{array}{l} F_\tau(\partial\Omega) = (I - K_\tau)(\partial\Omega) \neq 0, \ \forall \tau \in [0,1] \\ \deg(F_\tau, \Omega, 0) = \deg(F_0, \Omega, 0) \neq 0, \ \forall \tau \in [0,1] \end{array}$$

Finally, one checks easily that the *linear* problem F_0 has a unique solution in X, so that $\deg(F_0, \Omega, 0) \neq 0$. Therefore we have also $\deg(F_1, \Omega, 0) \neq 0$ which implies the existence of a solution to the problem in the bounded domain $[-a, a]$.

Remark 5. If the functions f and/or g have a discontinuity at θ_v and/or at θ_i, then we can prove the same result by using regularisation techniques in order to overcome the fact that K_τ is no longer continuous (see [4]).

2.1. Uniform *a priori* estimates. Proof of Lemma 4.

The aim of this section is the proof of Lemma 4. Without loss of generality, we can assume that $\tau = 1$. This lemma can be divided into two propositions:

Proposition 6. *[Estimates on the solutions]For every solution (T, Y, R, c) of problem (3) with $c \geq 0$, the following holds:*

$$\begin{array}{ccc} & c > 0, & \\ 0 \leq R \leq R_u, & 0 < T \leq T_b, & 0 \leq Y < T_b, \\ R' \leq 0, & 0 < T' \leq c T_b, & -\frac{c}{d} T_b \leq Y' \leq \frac{c}{d} T_b + \frac{4}{3} \pi n_u R_u^{3/2}. \end{array}$$

Proof. The estimates on the size R of the droplets are obvious. The estimates on the other unknowns follow the same lines as in [4]. □

Proposition 7. *[Estimates on the velocity]Let $h(s) = (T_b - s)\, f(s)$, and assume that $H(T_b) = \int_{\theta_i}^{T_b} h(s)\, ds < \infty$. Then, if (T, Y, R, c) is any solution of (3), c satisfies:*

$$\min\left(\frac{2}{d T_b^2} H(T_b), c_s^2\right) \leq c^2 \leq \max\left(\frac{\ln\left(\frac{2T_b}{\theta_v}\right)}{a_0}, \max\left(2M, \sqrt{\frac{2 M T_b}{\theta_v}}\right)\right) \tag{4}$$

□

We prove successively lower and upper-bounds independent of the size of the domain for the velocity c:

Proof of the Lower-Bound of Proposition 7: The simple but important alternative stated in Lemma 8 below allows us to take advantage of the estimates of Proposition 9 below:

Lemma 8. *[An important alternative] If $c \leq c_s := \sqrt{\frac{1}{R_{\mathrm{ui}}} \int_{\theta_v}^{\theta_i} \frac{g(s)}{s}\, ds}$, then $R(x_i) \leq 0$ with x_i satisfying $T(x_i) = \theta_i$.*

Proposition 9. *Every solution (T, Y, R, c) of problem (3), with $c \geq 0$, satisfy on $[-a, a]$:*

$$\begin{array}{l} |T + Y - T_b| \leq \left|\frac{1-d}{d}\right| (T_b - T) + \frac{4}{3} \pi n_u R^{3/2}(x) \\ |T + dY - T_b| \leq |1 - d| (T_b - T) + \frac{4}{3} \pi n_u R^{3/2}(x) \end{array} \tag{5}$$

and consequently

$$\begin{array}{lccccc} \mathrm{d} > 1: & \frac{1}{d}(T_b - T) - \frac{4}{3}\pi n_u R^{3/2}(x) & \leq & Y & \leq & (T_b - T) + \frac{4}{3}\pi n_u R^{3/2}(x) \\ 0 < \mathrm{d} < 1: & (T_b - T) - \frac{4}{3}\pi n_u R^{3/2}(x) & \leq & Y & \leq & \frac{1}{d}(T_b - T) + \frac{4}{3}\pi n_u R^{3/2}(x) \end{array} \tag{6}$$

Proof of Lemma 8:

Proof. (of Lemma 8) First, we recall that $T(x) = \theta_v e^{cx}$ on $[-a, x_i]$, so that $T'(x) = c\,T(x)$. The equation for R is:

$$\frac{\mathrm{d}R}{\mathrm{d}x} = \frac{\mathrm{d}R}{\mathrm{d}T}\frac{\mathrm{d}T}{\mathrm{d}x} = -\frac{1}{c}\,g(T(x)). \tag{7}$$

If we integrate $\frac{\mathrm{d}R}{\mathrm{d}T}$ between the temperatures θ_v and θ_i, we obtain:

$$R(x_i) - R_u = -\frac{1}{c^2}\int_{\theta_v}^{\theta_i} \frac{g(s)}{s}\,d\,s,$$

and, for $c \le c_s$, the droplets vaporise before they reach the reaction zone, so that the combustion is monophasic. □

Proof. (of Proposition 9)The proof follows the same lines as in [4]. It relies on suitable integrations of the auxiliary functions $W = T + Y$ and $Z = T + d\,Y$. We refer to [4] as well as the next section for an exposition of this classical manipulations. □

Now, we can use the same procedure as in Berestycki, Nicolaenko and Scheurer ([4]) in order to prove lower-bounds for the velocity c.

Proof. (of the lower-bound of Proposition 7) We give the details for the case $d > 1$ only, the proof for the case $d < 1$ being identical. First, if $c \ge c_s$, then the first inequality in (4) is proved. Consider now the case where $c \le c_s$, which implies that the combustion . We start now with the identity obtained by integration of $-T'' + c\,T' = f(T)\,Y$ between 0 and a, after multiplication by T':

$$-\frac{1}{2}|T'(a)|^2 + \frac{1}{2}\,c^2\theta_v^2 + c\int_0^a |T'(s)|^2\,d\,s = \int_{x_i}^a f(T(s))\,T'(s)\,Y(s)\,d\,s. \tag{8}$$

But, $c \le \langle \mathsf{cs} \rangle$, so that $R(x) = 0$, $\forall x \ge x_i$, and we deduce: $Y(x) \ge \frac{1}{d}\,(T_b - T(x))$, $\forall x \ge x_i$. We use this lower-bound for Y in the previous integral to obtain:

$$-\frac{1}{2}|T'(a)|^2 + \frac{1}{2}\,c^2\theta_v^2 + c\int_0^a |T'(s)|^2\,d\,s \ge \frac{1}{d}\int_{x_i}^a f(T(s))\,(T_b - T(s))\,T'(s)\,d\,s.$$

We make the change of variable $t := T(s)$ in the integral and obtain:

$$-\frac{1}{2}|T'(a)|^2 + \frac{1}{2}\,c^2\theta_v^2 + c\int_0^a |T'(s)|^2\,d\,s \ge \frac{1}{d}\,H(T_b).$$

Then, similar L^2 estimates can be obtained by integrating against T the equation for the temperature. This allows to control the integral above and therefore to prove that $\frac{2}{dT_b^2}\,H(T_b) \le c^2$, which is the desired result. □

Proof of the Upper-Bound of Proposition 7 We have the

Proposition 10. *Let* $M = \sup\limits_{[0, T_b]} f(s)$ *and* $a_0 > 0$ *fixed: then, for each* $a \ge a_0$, *any solution* (T, Y, R, c) *of (3) satisfies:*

$$c \le \max\left(\frac{\ln\left(\frac{2T_b}{\theta_v}\right)}{a_0}, \max\left(2\,M, \sqrt{\frac{2\,M T_b}{\theta_v}}\right)\right). \tag{9}$$

Proof. We compare the solution T of (3) to the solution $\bar{T}$ of the following system:

$$\begin{cases} -\bar{T}'' + c\bar{T}' = T_b M\,H(x) & o\,n] - a, a[\\ -\bar{T}'(-a) + c\bar{T}(-a) = 0, & \bar{T}(a) = T_b. \end{cases} \tag{10}$$

where H is the Heaviside function. By a direct calculation, we have:

$$\begin{cases} \bar{T}(x) = T(x), & on\ [-a,0], \\ \bar{T}(x) = T_b e^{cx}\left(\dfrac{M}{c^2}(1-e^{-ca}) + (1-\dfrac{M}{c}a)\, e^{-ca}\right), & on [0,a]. \end{cases} \tag{11}$$

By using maximum principle, we have $T \leq \bar{T}$ on $[-a,a]$. In particular, for $x=0$, we have:

$$\theta_v \leq T_b e^{-ca}(1-\frac{M}{c}) + T_b \frac{M}{c^2}(1-e^{-ca}). \tag{12}$$

We set $c_0 = \max(2\,M, \sqrt{\frac{2\,\mathrm{MT}_b}{\theta_v}})$, and we compute an upper bound of c independent of a. If $c \leq c_0$, the result is proved. Conversely, if $c > c_0$, then since $\theta_v \leq T_b e^{-ca} + \frac{\theta_v}{2}$, we have that $e^{ca} \leq \frac{2T_b}{\theta_v}$, and we conclude that:

$$c \leq \frac{\ln \frac{2T_b}{\theta_v}}{a_0}.$$

This completes the proof of the proposition. □

3. LARGE ACTIVATION ENERGY LIMIT

We deal now with the most important part of the paper. We want to derive an asymptotic model for the spray flame corresponding to the so-called large activation energy limit, where one assumes that the typical time associated to the chemical reaction is infinitely small compared to any other time-scales, namely in our case time-scales associated to the diffusion process or that associated to the vaporisation process. The most important result is that we are able to prove and quantify in the limit the existence of two different regimes for the propagation of the spray flame, which is a striking and novel feature compared to classical gas flames. On the one hand, when the radius of the droplets in the fresh mixture is small enough, the velocity of the flame is that of the equivalent gas flame, that is the correponding gas flame where all the fuel is injected in gaseous form. On the converse, above a critical radius of the incoming droplets, the velocity of the flame starts decreasing and, following the terms of Suard, Nicoli and Haldenwang ([25]), we are in the presence of a "Vaporisation Controlled" combustion regime. Moreover, the asymptotic analysis performed below provides details on the structure of the internal combustion layer in the presence of vaporising droplets. This are new mathematical results, consistent with the results and numerical observations made by the previous authors.

After stating the results, Section 3.0.1 is devoted to a heuristic proof of the limiting profile. It allows us to emphasize the existence of the two combustion regimes and to give more insight in the system. The rigourous proof together with the study of the internal layer are done in Section 3.1.

We start again with system (1) in which the reaction term is now set as:

$$f_\varepsilon(T) := \begin{cases} 0, & T < \theta_i, \\ \dfrac{1}{\varepsilon^\gamma}\, exp\left(\dfrac{T-T_b}{\varepsilon}\right), & T > \theta_i. \end{cases} \tag{13}$$

and we are interested in the existence of travelling waves in the limit where ε goes to 0. Formally, one expects at the limit that the reaction term occurs in an infinitely small region, that we set at the origin $x=0$.

$$\begin{cases} -T'' + c\,T' &= cT_b\delta_{x=0}, \\ -d\,Y'' + c\,Y' &= -cT_b\delta_{x=0} + 2\,\pi\, n_u\, g(T)\sqrt{R}, \\ c\,R' &= -g(T)\, H(R). \end{cases} \tag{14}$$

We want to prove:

Theorem 11. *Let $M_u := 4/3\,\pi\,n_u\,R_u^{3/2}$ fixed be given, such that $T_b = Y_u + M_u$ is kept constant when R_u varies. Let γ be given in the intervall $[0,2]$. We have that*

- $*$. *If $\gamma < 2$ then the solutions $(T_\varepsilon, Y_\varepsilon, R_\varepsilon, c_\varepsilon)$ to system (1) tend to a degenerate travelling wave in the sense that c_ε tends to 0.*
- $*$. *If $\gamma = 2$ then the solutions $(T_\varepsilon, Y_\varepsilon, R_\varepsilon, c_\varepsilon)$ to system (1) tend to a travelling wave solution of the limiting system (14) above, were the limiting velocity c has the expression*

$$c = \min\left(\frac{1}{T_b}\sqrt{\frac{2}{d}} \quad , \quad \sqrt{\frac{1}{R_u}\int_{\theta_v}^{T_b} \frac{g(s)}{s}\,d\,s}\right) \tag{15}$$

As a consequence, the behaviour of the velocity c of the limiting travelling waves shows a bifurcation when the radius R_u of the incoming droplets is greater than the critical radius R_c:

$$R_c := \frac{d\,T_b^2}{2}\int_{\theta_v}^{T_b} \frac{g(s)}{s}\,d\,s \tag{16}$$

- $*$. *Moreover, when $\gamma = 2$ and $R_u \geq R_c$, then the size of the vaporising zone intersecting the reaction zone is at most of order $O(\varepsilon^{2/5}|\ln\varepsilon|)$ for small values of $\varepsilon > 0$.*

□

3.0.1. Heuristic asymptotic analysis of spray flames.

Starting from Hypothesis 1&2 below, we show in this section how to solve completely the spray flame problem using simple arguments. All these hypotheses and statement become rigourous in the next section. In this study, the main parameter is the size of the incoming droplet. When this parameter varies, it is crucial to keep constant the temperature T_b of the burnt gases. Equivalently, this means that we want to compare flames whose total amount of (liquid and/or gaseous) incoming fuel is the same, which is a natural constraint. The velocity, directly proportional to the burning rate, therefore measures the efficiency of the combustion. For example, the total mass $M_u := 4/3\,\pi\,n_u\,R_u^{3/2}$ of incoming liquid fuel is kept constant by adjusting the number density n_u of the incoming droplets when R_u varies.

Hypothesis 1: Combustion occurs instantaneously at the flame front $x_f := 0$ at the temperature T_b of the burnt gases.

Hypothesis 2: For small enough sizes of the incoming droplets, the vaporisation finishes strictly before the reaction zone.

Here x_f is the position of the flame front, taken as the origin, x_v is the position of the beginning of the vaporisation zone, i.e. $T(x_v) = \theta_v$, and x_{vf} is the position of the vaporisation front, that is the first point where the size of the droplets vanishes. For small droplets, Hypothesis 2 implies that the velocity of the wave is equal to the Zeldovich velocity $c_m := \sqrt{2/d}/T_b$ of the classical large activation energy monophasic gas flame. Indeed, the internal structure of the flame involves only the unknowns T and Y as well as the corresponding profiles in the hot gases, all of which are not affected by the presence of the droplets. Increase now the size of the droplets, keeping M_u constant. Assuming that the corresponding profiles vary continuously with respect to the size of the droplets, define $R_c > 0$ as the critical smallest size at which the droplets finish vaporising exactely at the flame front. This means in particular that for all $R_u \in]0, R_c[$, the velocity of the flame is equal to c_m. We have:

Statement. Assume for simplicity that the vaporisation law is the step-function $g(T) = g_0\,H\,(T - \theta_v)$. Let $R_c > 0$ be the critical size of droplets defined above. Then we have:

$$R_c = \frac{g_0}{c_m^2}\ln\left(\frac{T_b}{\theta_v}\right) \tag{17}$$

□

Indeed, according to the vaporisation law, for a given size R_u of incoming droplets, the quantities $c(R_u)$, $x_v(R_u)$ and $x_{vf}(R_u)$ obey the relation:

$$x_{vf} = x_v + \frac{c}{g_0} R_u = \frac{1}{c} \ln(T_b \theta_v) + \frac{c}{g_0} R_u \tag{18}$$

Then, by continuity of the solution with respect to the parameter R_u, on the one hand we must have $c(R_c) = c_m$ and on the other hand, the definition of R_c implies $x_{vf}(R_c) = 0$. These two relations give the relation for R_c in the Statement above.

Now, what happens now if $R_u > R_c$? We have:

Statement. In the case where $R_u > R_c$, we have necessarily $x_{vf}(R_u) = 0$ and therefore

$$c(R_u) = c_d(R_u) := \sqrt{\frac{g_0}{R_u} \ln\left(\frac{T_b}{\theta_v}\right)} \tag{19}$$

□

where c_d designates the velocity of the flame in the diphasic case as opposed to the monophasic case abovementionned. The second part of the statement is a direct consequence of the first part, after solving the equation $x_{vf}(R_u) = 0$ for c. Then, it is impossible that $x_{vf} > 0$. Otherwise, vaporisation, combustion and finally an elevation of temperature would occur after the flame front, which is prohibited by the fact that the temperature is already equal to its maximum value at $x = 0$. On the converse, $x_{vf}(R_u) < 0$ is impossible. Indeed, one checks easily that we would have $x_{vf}(R) < 0$ for all $R < R_u$, a contradiction with $x_{vf} R_c = 0$.

As a conclusion, for any R_u, we have the alternative:

Statement. Let R_u be given. Then, the limiting system is completely determined in terms of the following alternative involving the data of the problem:

- $R_u \leq R_c$ is equivalent to $\{x_{\mathrm{vf}}(R_u) < 0$ and $c(R_u) = c_m\}$
- $R_u > R_c$ is equivalent to $\{x_{\mathrm{vf}}(R_u) = 0$ and $c(R_u) = c_d(R_u) \leq c_m\}$

□

As announced, we observe that the velocity of the spray-flames diminishes for size of the incoming droplets larger than a critical value. This change of regime occurs abruptly at that critical value. We recover here the existence of the so-called "vaporisation-limited" regime emphasized in [HNS]. In particular, it follows from these remarks that substituting some gas fuel for some liquid fuel in the fresh gas can only lead to a decrease of the burning rate, i.e. the velocity, of the flame (at least in the scope of the hypotheses of this Section). If one sets the origin $x = 0$ in the fresh gas where $T(x = 0) = T_0 < T_b$, then the decrease of the velocity of the flame means an increase of the preheating zone. In other words, large droplets push the reaction front towards the hot gases.

We invite the reader to verify that we would obtain exactely the same conclusions by replacing Hypothesis 2 above by the following:

Hypothesis 3: The velocity of the spray flame is always lower or equal to the velocity c_m of the classical gas-flame having the same temperature in the burnt gas.□

The rigourous analysis of the structure of the spray flame in the large activation energy limit precisely aims at proving Hypotheses 1 and 3 plus some more information about the structure of the internal combustion layer.

3.1. Large activation energy values : existence of flames and their structure.

In the rigourous proof of the convergence of the system to a limiting travelling wave, the main difficulty is to obtain uniform estimates from below as well as from above for the velocity. Thanks to the results of the previous section, this directly yields a solution to the limiting system in the form (14) above. In order to perform these estimates, one has to analyse the internal structure of the reaction zone. Following Zeldovich, this is done by introducing scaled variables and coordinates as we shall see below. First notice that the choice of the value $T(0) := T_b - A\varepsilon \ln\varepsilon$ provides an information about the velocity of the flame in terms of the temperature profile at the origin:

First expression of the velocity of the flame

Thanks to the remark that, for A large enough, one has $f_\varepsilon(T_\varepsilon)Y_\varepsilon = O(\varepsilon^{A/2})$ whenever $T_\varepsilon \leq T_b + A\varepsilon\ln\varepsilon$, it follows that $T_\varepsilon(x) = (T_b + A\varepsilon\ln\varepsilon)\, e^{c_\varepsilon x} + O(\varepsilon^{A/2})$ on $]-\infty, 0]$. Integrating the equation for T_ε between $-\infty$ and 0:

$$-T_\varepsilon'(0) + c_\varepsilon(T_b + A\varepsilon\ln\varepsilon) = \int_{-\infty}^0 f_\varepsilon(T_\varepsilon)Y_\varepsilon, \tag{20}$$

where the Right-Hand-Side is $O(\varepsilon^{A/2})$, we have therefore

$$c_\varepsilon = \frac{T_\varepsilon'(0)}{T_b} + O(\varepsilon\ln\varepsilon). \tag{21}$$

It remains to estimate the derivative of the temperature profile at the origin, which we do by studying the system on the intervall$[0, +\infty[$.

Notations. Streched system.

The boiling point is denoted by $x_{v,\varepsilon}$ and satisfies $T_\varepsilon(x_{v,\varepsilon}) = \theta_v$, the ignition point is denoted by $x_{i,\varepsilon}$ and satisfies $T_\varepsilon(x_{i,\varepsilon}) = \theta_i$, the end of vaporisation occurs at $x_{\mathrm{vf},\varepsilon}$ satisfies $R_\varepsilon(x_{\mathrm{vf},\varepsilon}) = 0$, and we recall that $T_\varepsilon(0) = T_b + A\varepsilon\ln\varepsilon$. Also, we use the auxiliary functions $W_\varepsilon := T_\varepsilon + Y_\varepsilon - T_b$ and $Z_\varepsilon := T_\varepsilon + dY_\varepsilon - T_b$. Now, the exponent $\gamma \leq 2$ being given, we define the streched variables by setting:

$$\xi = \frac{x}{\varepsilon^{\gamma/2}},\ p_\varepsilon(\xi) = \frac{T_\varepsilon(x) - T_b}{\varepsilon},\ q_\varepsilon(\xi) = \frac{Y_\varepsilon(x)}{\varepsilon},\ w_\varepsilon(\xi) = \frac{W_\varepsilon(x)}{\varepsilon},\ z_\varepsilon(\xi) = \frac{Z_\varepsilon(x)}{\varepsilon}. \tag{22}$$

$$p_\varepsilon(0) = A\ln\varepsilon, \quad p_\varepsilon(\infty) = 0, \quad q_\varepsilon(\infty) = 0, \quad w_\varepsilon(\infty) = 0, \quad z_\varepsilon(\infty) = 0$$

Moreover, we naturally denoted here $(\xi_{v,\varepsilon},\ \xi_{i,\varepsilon},\ \xi_{\mathrm{vf},\varepsilon}) = (x_{v,\varepsilon},\ x_{i,\varepsilon},\ x_{\mathrm{vf},\varepsilon})/\varepsilon^{\gamma/2}$. Finally, it is important to recast the expression of the velocity in terms of the streched system, namely, for A fixed large enough:

$$c_\varepsilon := \varepsilon^{1-\gamma/2}\frac{p_\varepsilon'(0)}{T_b} + O(\varepsilon\ln\varepsilon). \tag{23}$$

and it remains to estimate the derivative of the temperature profile in the streched variables.

First Estimate of $p_\varepsilon'(0)$

The estimate of $p_\varepsilon'(0)$ goes in two steps. First, we prove here that this quantity is bounded uniformly with respect to ε. We provide an optimal value of the upper-bound later.

It is convenient to write the system in terms of the unknowns $(p_\varepsilon, w_\varepsilon)$ or $(p_\varepsilon, z_\varepsilon)$ defined above. Simple algebra yields:

$$\begin{cases} -p_\varepsilon'' + c_\varepsilon e^{\gamma/2} p_\varepsilon' = (w_\varepsilon - p_\varepsilon)\, e^{p_\varepsilon} = -\dfrac{1}{d}\,(z_\varepsilon - p_\varepsilon), \\ -w_\varepsilon'' + c_\varepsilon \varepsilon^{\gamma/2} w_\varepsilon' = \left(\dfrac{1-d}{d}\right) c_\varepsilon \varepsilon^{\gamma/2} p_\varepsilon'' + \dfrac{2\,\pi\, n_u}{\sqrt{c_\varepsilon}} \varepsilon^{\frac{5\gamma}{4}-1} \sqrt{(\xi_{\mathrm{vf},\varepsilon} - \xi)_+}, \\ -z_\varepsilon'' + \dfrac{c_\varepsilon \varepsilon^{\gamma/2}}{d} z_\varepsilon' = \left(\dfrac{1-d}{d}\right) c_\varepsilon \varepsilon^{\gamma/2} p_\varepsilon' + \dfrac{2\,\pi\, n_u}{\sqrt{c_\varepsilon}} \varepsilon^{\frac{5\gamma}{4}-1} \sqrt{(\xi_{\mathrm{vf},\varepsilon} - \xi)_+}. \end{cases} \tag{24}$$

First, integrating the equation for the temperature against p_ε' gives:

$$c_\varepsilon \varepsilon^{\gamma/2} \int_0^\infty (p_\varepsilon')^2\, d\,\xi + \frac{(p_\varepsilon'(0))^2}{2} = \int_0^\infty (w_\varepsilon - p_\varepsilon)\, e^{p_\varepsilon} p_\varepsilon' d\,\xi = \frac{1}{d}\int_0^\infty (z_\varepsilon - p_\varepsilon)\, e^{p_\varepsilon} p_\varepsilon' d\,\xi \tag{25}$$

where the contribution of p_ε in the last two integrals can be integrated explicitely. It remains to estimate w_ε or z_ε in terms of p_ε. For that, we distinguish the two cases $0 < d \leq 1$ and $d \geq 1$.

Case $0 < d \leq 1$*:* An integration of the equation for w_ε together with the condition at infinity $w_\varepsilon'(+\infty) = 0$ yields, since the contribution of the vaporisation term is nonnegative:

$$\begin{aligned}
w_\varepsilon'(\xi) \;&\geq \frac{1-d}{d}\int_{\eta=\xi}^{\eta=+\infty} \exp\left(-c_\varepsilon \varepsilon^{\gamma/2}(\xi-\eta)\right) p_\varepsilon''(\eta)\,\mathrm{d}\eta \\
&= \frac{1-d}{d}\Big\{\left[\exp\left(c_\varepsilon \varepsilon^{\gamma/2}(\xi-\eta)\right) p_\varepsilon'(\eta)\right]_{\eta=\xi}^{\eta=+\infty} \\
&+ \int_{\eta=\xi}^{\eta=+\infty} \exp\left(c_\varepsilon \varepsilon^{\gamma/2}(\xi-\eta)\right) c_\varepsilon \varepsilon^{\gamma/2} p_\varepsilon'(\eta)\, d\,\eta\Big\} \\
&\geq \varepsilon\frac{1-d}{d}(-p_\varepsilon'(\xi))
\end{aligned} \tag{26}$$

because $0 < d \leq 1$ and the last integral is positive. Finally, since $w_\varepsilon(+\infty) = 0$, and $p_\varepsilon' \geq 0$, we have:

$$\forall \xi \geq 0, \quad w_\varepsilon(\xi) \leq |p_\varepsilon(\xi)| = -p_\varepsilon(\xi) \tag{27}$$

Hence, Eq. (25) gives

$$\frac{p_\varepsilon'(0)^2}{2} \leq \frac{1-d}{d} + 1 = \frac{1}{d} \tag{28}$$

Case $d \geq 1$*:* Now, a direct integration of the equation for z_ε yields, since the contribution of the vaporisation term is nonnegative:

$$z_\varepsilon'(\xi) - \frac{c_\varepsilon \varepsilon^{\gamma/2}}{d}\, z_\varepsilon(\xi) \geq \frac{d-1}{d}\, c_\varepsilon \varepsilon^{\gamma/2} p_\varepsilon(\xi) \tag{29}$$

Hence

$$\begin{aligned}
z_\varepsilon(\xi) \;&\leq \frac{d-1}{d}\int_{\eta=\xi}^{\eta=+\infty} \exp\left(\frac{c_\varepsilon \varepsilon^{\gamma/2}}{d}(\xi-\eta)\right) c_\varepsilon \varepsilon^{\gamma/2}(-p_\varepsilon(\eta))\mathrm{d}\eta \\
&\leq \frac{d-1}{d}(-p_\varepsilon(\xi))\int_{\eta=\xi}^{\eta=+\infty} c_\varepsilon \varepsilon^{\gamma/2} \exp\left(\varepsilon\frac{c_\varepsilon \varepsilon^{\gamma/2}}{d}(\xi-\eta)\right) d\,\eta \\
&= (d-1)(-p_\varepsilon(\xi))
\end{aligned} \tag{30}$$

where we have used the monotonicity of p_ε. Hence, Eq. (25) gives

$$\frac{p_\varepsilon'(0)^2}{2} \leq \frac{(d-1)+1}{d} = 1 \tag{31}$$

As a conclusion, $p_\varepsilon'(0)$ is uniformly bounded with respect to ε and the boundedness of c_ε follows from (23) as well as:

Lemma 12. *Let $\gamma < 2$ be given. Then, the velocity c_ε tends to zero as ε goes to zero.*

Hence, in the rest of this section, we shall only consider the case where $\gamma = 2$.

Approximate System: Come back now to the unknowns $(p_\varepsilon, z_\varepsilon)$. It is convenient to introduce the approximate problem:

$$\begin{cases}
-\tilde{p}_\varepsilon'' &= \dfrac{1}{d}\,(\tilde{z}_\varepsilon - \tilde{p}_\varepsilon)\, e^{\tilde{p}_\varepsilon}, \\
-\tilde{z}_\varepsilon'' &= \dfrac{2\,\pi\, n_u}{\sqrt{c_\varepsilon}} \varepsilon^{3/2}\sqrt{(\xi_{\mathrm{vf},\varepsilon} - \xi)_+},
\end{cases} \tag{32}$$

with the same boundary conditions as for $(p_\varepsilon, z_\varepsilon)$, but where we omitted the convection terms involving the factor $c_\varepsilon \varepsilon$. Classical results on elliptic regularity ensure that $(p_\varepsilon, z_\varepsilon) - (\tilde{p}_\varepsilon, \tilde{z}_\varepsilon)$ as well as their derivatives $(p_\varepsilon', z_\varepsilon') - (\tilde{p}_\varepsilon', \tilde{z}_\varepsilon')$ are of order ε, uniformely on $[0, +\infty[$, in the space of continuous functions.

Estimate of the Vaporisation Front $\xi_{\mathrm{vf},\varepsilon}$

It is now possible to estimate the size $\xi_{\mathrm{vf},\varepsilon}$ of the intervall of the vaporisation zone intersecting that of the reaction zone (i.e. $\xi > 0$):

Lemma 13. *There exists a constant $C>0$ such that for ε small enough:*

$$\xi_{\mathrm{vf},\varepsilon} \leq C\varepsilon^{-3/2}|\ln\varepsilon|^{2/5}, \quad \text{or equivalently} \quad x_{\mathrm{vf},\varepsilon} \leq C\varepsilon^{2/5}|\ln\varepsilon|^{2/5} \tag{33}$$

Proof. Integrate the equation for $\tilde{z}_\varepsilon$ taking into account the vaporisation term. For that, recall that we have at the vaporisation front $\tilde{z}_\varepsilon(\xi_{\mathrm{vf},\varepsilon})=0$ and $\tilde{z}_\varepsilon'(\xi_{\mathrm{vf},\varepsilon})=0$. It comes:

$$\tilde{z}_\varepsilon'(\xi) = C\frac{\varepsilon^{3/2}}{c_\varepsilon^{1/2}}(\xi_{\mathrm{vf},\varepsilon}-\xi)^{3/2} \geq 0 \tag{34}$$

where C denotes a generic positive constant independent of ε. Integrating again, we have immediately, since $z_\varepsilon(+\infty)=0$,

$$\tilde{z}_\varepsilon(\xi) = -C\frac{\varepsilon^{3/2}}{c_\varepsilon^{1/2}}(\xi_{\mathrm{vf},\varepsilon}-\xi)^{5/2} \leq 0 \tag{35}$$

Recalling that $z_\varepsilon = \tilde{z}_\varepsilon + O(\varepsilon)$, we have

$$-C\frac{\varepsilon^{3/2}}{c_\varepsilon^{1/2}}\xi_{\mathrm{vf},\varepsilon}^{5/2} + O(\varepsilon) = z_\varepsilon(0) = p_\varepsilon(0) + d\,q_\varepsilon(0) \geq A\ln\varepsilon \tag{36}$$

because $q_\varepsilon(0)\geq 0$, and the result follows thanks to the boundedness of c_ε. □

Second Estimate of $p_e'(0)$

We precise here the upper-bound and give a lower-bound for the velocity and prove:

Lemma 14. *Let d be given. We have for a certain $\underline{c}>0$, and for ε small enough:*

$$\frac{\underline{c}^2T_b^2}{2} \leq \frac{(p_\varepsilon'(0))^2}{2} \leq \frac{1}{d} + 0(\varepsilon^{2/5}|\ln\varepsilon|) \tag{37}$$

Proof. Since $\tilde{z}_\varepsilon$ above is negative, it follows that $z_\varepsilon \leq O(\varepsilon)$, so that we have from (25) the uniform upper-bound:

$$\forall d, \quad \frac{p_\varepsilon'(0)^2}{2} \leq \frac{1}{d} + O(\varepsilon) \tag{38}$$

This inequality becomes an equality in the case where $\limsup_{\varepsilon\to 0}\xi_{vf,\varepsilon}<0$ and we have therefore in that case a lower-bound for the velocity of the limiting flame. This happens in particular in the case where $\limsup_{\varepsilon\to 0} x_{vf,\varepsilon}<0$ (which implies $\xi_{vf,\varepsilon}\to -\infty$) that is when the droplets finish vaporising strictly *before* the reaction front $x=0$. This remark allows us to use the same alternative (see Lemma 8) as in the previous section. Namely, using the notations of the heuristic analysis, choose $0<\underline{c}<c_d(R_u)$. Then, for ε small enough, $c_\varepsilon \leq \underline{c}$ implies that the droplets finish vaporising strictly before the reaction front. Hence, $c_\varepsilon \simeq c_m$. This completes the proof. □

3.2. Large activation energy values : Limiting system as $\varepsilon\to 0$.

It remains to prove that the limiting system in the High Energy Activation limit is indeed system (14) with the velocity c as in Theorem 11.

Proof. (of Theorem 11 – Continued) Following [4], the uniform smoothness in $\mathcal{C}^1(\mathbb{R})$ of the sequence T_ε together with the uniform lower-bound for c_ε allow us to extract converging subsequences $T_{\varepsilon'}$, $x_{vf,\varepsilon'}$, $x_{v,\varepsilon'}$, $c_{\varepsilon'}$ on any large enough compact intervall around the origin. Then, let $x_{vf,1}\leq 0$ and $x_{vf,2}\leq 0$ be two limits of subsequences of the sequence of vaporisation fronts $x_{vf,\varepsilon}$. If $x_{vf,1}<x_{vf,2}<0$ then necessarily we have for the corresponding velocities $c_1=c_m=c_2$, which implies $x_{vf,1}=x_{vf,2}$, a contradiction. Now, recalling the definition of R_c in the heuristic analysis, if $x_{vf,1}<x_{vf,2}=0$, we must have both $R_u<R_c$ and $R_u\geq 0$, a contradiction again. As a conclusion, we have necessarily $x_{vf,1}=x_{vf,2}$, hence also $c_1=c_2$, so that the limiting profile is unique, and all the sequence of solutions tend towards their respective limits. □

4. EXTENSIONS

4.1. Singular limits for vaporisation and/or combustion.

The case of rapid vaporisation: There are situations where the vaporisation of the droplets occurs in a very small region compared to the size of the heating zone, that is situations where the time for complete vaporisation is very small. This may happen in two situations. First, when the droplets are very small, we have $R_u = \eta_1 R_{u,0}$ where η_1 is a small parameter and $R_{u,0}$ is a reference size of order 1. In that case, the density number of droplets in the fresh gases is normalised to $n_u = n_{u,0}/\eta_1^{3/2}$. This allows us as usual to maintain a constant temperature in the burnt gases $T_b = Y_u + 4/3\,\pi\, n_{u,0}\, R_{u,0}^3/2$ while letting η_1 go to 0. Second, when the vaporisation rate is very large, me may normalize it as $g(T) = g_0(T)/\eta_2$ where η_2 is a small parameter. Recall here that $g_0(T)$ is an increasing function, positive for temperatures larger than θ_v. We have:

Set the origin as $x_v = 0$, such that $T(0) = \theta_v$. Then, when η_1 and/or η_2 tend to 0, the solution $(T, Y, R, c)_{\eta_1,\eta_2}$ of problem (1) converges to a solution of the following system, where complete vaporisation occurs instantaneously at $x_v = 0$:

$$\begin{cases} -T'' + c\,T' = f(T)Y \\ -d\,Y'' + c\,Y' = -f(T)Y + c\dfrac{4}{3}\pi n_1 R_{u,1}^{3/2}\delta(x=0) \end{cases}$$

with the usual boundary conditions

Proof. The details of the proof are left to the reader. We refer to [6] for more details. □

The case of rapid vaporisation and combustion: The previous limit is easily combined whith the high activation energy asymptotics. In that case, when rapid vaporisation and reaction occur at the same time, the limiting system has two singular terms. Both the vaporisation and the reaction term are concentrated in infinitely small regions. This is in some sense a simplified situation where vaporisation necessarily ends before the reaction occurs, since $\theta_v < T_b$. Hence, the velocity of the limiting wave is obviously $c = c_m$.

4.2. Results for a polydisperse spray.

All the previous results can be extended in the case of polydisperse sprays, that is in the case where droplets of possibly different sizes are present at a given position. In that case, we need a statistical description of the distribution of size of the droplets as was first pointed out by Williams [?, 26]. For our problem, we assume more precisely that the distribution of size of droplets is known in the fresh gases. Introduce for that the probability density number $h_u(R)$ of droplets in the fresh gases, such that $h_u(R)\,d\,R$ is the probable number of droplets having size in the range $[R, R + d\,R]$. We define in the same manner the density number $h(x, R)$ at any position x and we have the limiting condition in the fresh gas $h(-\infty, R) = h_u(R)$. Also, we assume that h_u is continuous on $\mathbb{R}_+$ and has compact support. Then, the spray flame problem writes:

$$\begin{cases} -T'' + c\,T' & = Y f(T), \\ -d\,Y'' + c\,Y' & = -Y f(T) + 2\,\pi\, g(T) \displaystyle\int_R R^{1/2}\, h(x,R)\, d\,R, \\ c\partial_x h + \partial_R (h(-g(T)\,H(R)) & = 0, \end{cases} \tag{39}$$

In this setting, the total mass fraction M_u of liquid fuel in the fresh gas, and therefore the corresponding temperature T_b of the burnt gases are given by:

$$M_u = \int_0^{+\infty} \frac{4}{3}\pi\, R^{3/2}\, h_u(R)\, d\,R, \quad T_b = Y_u + M_u \tag{40}$$

Notice finally that a monodisperse spray corresponds to the situation where h_u writes as:

$$h_u(R) = n_u\, \delta\,(R - R_u) \tag{41}$$

Existence of Travelling Waves: The proof of the existence of a travelling wave for the polydisperse spray follows exactely the same lines as in the monodisperse case. In the alternative stated in Lemma 8, page 4, the threshold velocity may now be written in terms of $R_m := \sup\{R; h_u(R) > 0\}$ instead of R_u.

Asymptotic Limits: The limiting profiles are again of two types, but now the critical radius R_c must be compared to R_m defined above in order to know if the regime of combustion is controlled by vaporisation or not. Also, in the case of infinitely rapid vaporisation, the vaporisation eventually concentrates in an infinitely small region. Hence, there is no difference between a monodisperse and polydisperse spray in this asymptotics. We refer the reader to [6] for more details.

4.3. Other geometries.

The large activation energy limit analysis can be carried out in other onedimensionnal geometries, such as the anchored flame on the half-line, or counter-flow-like configurations, or finally radial in dimension 2 or 3. In each case, we are able to solve explicitely the problem (provided the expression of the vaporisation law is relatively simple), and it is possible to study the influence of different parameters on the existence of a profile or on the value of the burning rate for example. Relevant parameters are the typical velocities of injection of the gas and/or droplets, the value of the vaporisation rate, the dimension. We refer the reader to [6] for more details.

5. Perspectives and Concluding Remarks

This paper considered the existence of travelling fronts for a simple onedimensionnal thermodiffusive lean spray flame model. We proved the existence of travelling waves for general combustion and vaporisation laws. As far as qualitative results are concerned, the most important part of the paper is the study of the high activation energy limit for the system. Indeed, the limiting problem involve simple explicit profiles but still retains the important features of the dynamics. Extensions of these results to the cases of fast vaporisation, polydisperse sprays or other onedimensionnal geometries was briefly mentionned. The corresponding details can be found in [6].

The present work is therefore a first step towards the rigorous derivation of asymptotic models for spray flames. However, much remains to be done in order to understand the effect of droplets on the dynamics of the flame structure, as observed by many physicists and experimentalists (see [14] and the references therein). First, as already noticed in the Introduction, the model under consideration in this paper omits the effects of the latent heat. It would be very interesting to derive some high activation energy models incorporating these effects, in the hope of deriving explicit expressions of the limiting profiles and combustion rates, as well as extinction limits. Second, a striking consequence of the high activation energy asymptotics is that droplets *cannot* cross the flame front in that limit, but may enter the reaction zone only for large but not infinite values of the activation energy. It would be interesting to derive intermediate asymptotic models making explicit the structure and thickening of the combustion region for large but not infinite .

As far as dynamical phenomena are concerned, the problem of the stability of spray flame system are of particular importance. As a first step, one has to study the nonlinear stability for small perturbations of the thermodiffusive lean spray flame model (see [6, 5]). As a second step, we are interested in the problem of acoustic instabilities in spray flame systems. In this direction, the work Clavin an Sun [7] is of particular interest to us because these authors propose a framework for the mathematical analysis of gas or spray flames coupled to the acoustics. This mathematical analysis relies on the possibility of deriving explicit expressions of the solutions to the problem. This is only possible in some asymptotic limits. For that, the authors consider the large activation energy limit for combustion phenomena and assume an instantaneous vaporisation of the droplets. As a consequence, both the combustion and the vaporisation zones reduce to infinitely small regions, whose internal layer is analysed.

However, assuming that the vaporisation zone is very small compared to the preheating zone is a very restrictive assumption in many applications where the droplets can spread into the preheating zone, approach the combustion zone, or even cross the combustion zone. We have shown in the present paper that it is possible to handle analytically such situations where the droplets approach or reach the reaction front, and where the slowly vaporising droplets induce a dramatic change of the combustion rate. It would therefore be intersting to study mathematically and generalise the work of Clavin and Sun [7] to more general situations regarding the size of the vaporisation zone. It is reasonable to expect that different size of the vaporising region would give rise to different behaviours or stability properties of the system.

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